International Journal of Scientific & Engineering Research, Volume 7, Issue 11, November-2016
ISSN 2229-5518

On Some Results Similar to the Banach Steinhauss Theorem for Normed Linear Spaces

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Abstract
Let X and Y be normed linear spaces, $B(X,Y)$ the space of all bounded and linear maps from X to Y and $\{T_n\} \subseteq B(X,Y),\ n \geq 1$. We carry out a critical survey of the Banach Steinhauss theorem and present results similar to those presented in the theorem, under milder conditions and in the more general normed linear spaces.

Introduction

Let X and Y be normed linear spaces over a scalar field, f. A mapping $T : X \to Y$ is said to be linear if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),\ \forall\ x, y \in X$ and $\alpha,\ \beta \in f$.
A linear map $T : X \to Y$ is said to be bounded if there exists a real constant $k > 0$, such that $||Tx|| \leq k||x||,\ \forall x \in X$.

We recall that $B(X,Y) := \{T : X \to Y\ \textit{such that}\ T\ \textit{is linear and bounded}\}$. With pointwise addition and scalar multiplication defined respectively on $B(X,Y)$ by $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and $(\alpha T)(x) = \alpha T(x), \forall\ T, T_1, T_2 \in B(X,Y),\ x \in X$, and $\alpha \in f$, it is a routine exercise to verify that if $T_1, T_2 \in B(X,Y)$, then $-T_2 \in B(X,Y)$ and $T_1 - T_2 \in B(X,Y)$. Furthermore, if $T \in B(X,Y)$, then we define $||T||$ as $||T|| := inf\{k > 0 : ||Tx|| \leq k||x||\}$. With this, it is easily seen that $||T|| \leq k$ and $||Tx|| \leq ||T||||x||$.
The aim of this article is to present conditions different from those in the Banach Steinhauss Theorem (see e.g [1]) under which $sup||T_n|| < \infty$, where $\{T_n\} \subseteq B(X,Y),\ \forall n \geq 1$. We shall make use of the following definitions and theorems in the sequel.

Uniform Boundedness Principle (see e.g [1-7]): Let X and Y be normed linear spaces, over a scalar field, f. Let Δ be an arbitrary index set and $\{T_{\alpha\in\Delta}\} \subseteq B(X,Y)$. The uniform boundedness principle or theorem guarantees conditions under which $sup||T_\alpha|| < \infty, \forall \alpha \in \Delta$. The conditions are embodied in the following theorem:

Theorem 1 (Uniform Boundedness Theorem, see e.g [3]): Let X be a Banach space and Y be a normed linear space. Let $\{T_{\alpha\in\Delta}\} \subseteq B(X,Y)$, and suppose for each $x \in X$, there exists a real constant $M_x > 0$ such that $||T\alpha x|| \leq M_x,\ \forall\ \alpha \in \Delta$ (or equivalently $sup||T_\alpha x|| < \infty$), then $sup||T_\alpha|| < \infty, \forall \alpha \in \Delta$.

The Uniform Boundedness Theorem (UBT) is one of the corner stone theorems of Functional Analysis. The Banach Steinhauss Theorem (see e.g [1], [3]), which we shall state below is actually a consequence of the UBT.

Theorem 2 (Banach Steinhauss Theorem, see e.g [1]): Let X be a Banach space and Y be a normed linear. Let $\{T_n\} \subseteq B(X,Y)$. If for each $x \in X,\ \ T_n x \rightarrow Tx$ then:
(i) $sup||T_n|| < \infty$.
(ii) $T \in B(X,Y)$
(iii) $||T|| \leq liminf||T_n||$
We also recall the following definition:

Definition : Let X be a normed linear space. A sequence $\{x_n\} \subseteq X,\ n \geq 1$, is said to be Cauchy if for any $\epsilon > 0$ given, there exists an integer $N > 0$ such that $||x_n - x_m|| \leq \epsilon,\ \forall\ n,m \geq N$.
We now give some examples which investigate whether or not a sequence of bounded linear operators is Cauchy.

Example 1: Let $X = Y = R$ (the reals) with its usual norm. Define $T_n : R \rightarrow R$ by $T_n x = nx$. Then it is easily verifiable that $T_n \in B(X,Y)$, for each $n \geq 1$. Now for $n,\ m\ with\ n > m$, we have
$|(T_n - T_m)x| = |T_n x - T_m x| = |(n-m)x| = (n-m)|x|$. This implies
$||T_n - T_m|| \leq n - m < n \rightarrow \infty\ as\ n \rightarrow \infty$. Therefore $\{T_n\}$ is not a Cauchy sequence.

Example 2: Let $X = Y = R$ (the reals) with its usual norm. Define $T_n : R \rightarrow R$ by $T_n x = \frac{1}{n}x$. Then it is easily verifiable that $T_n \in B(X,Y)$, for each $n \geq 1$. Now for $n,\ m\ with\ m > n$, we have
$|(T_n - T_m)x| = |T_n x - T_m x| = |(\frac{1}{n} - \frac{1}{m})x| = (\frac{1}{n} - \frac{1}{m})|x|$. This implies
$||T_n - T_m|| \leq (\frac{1}{n} - \frac{1}{m}) < \frac{1}{n} \rightarrow 0\ as\ n \rightarrow \infty$. Therefore $\{T_n\}$ is a Cauchy sequence.

The foregoing two examples show that is is not a difficult task to determine whether a sequence $\{T_n\} \subseteq B(X,Y)$ is Cauchy or not. With this, it is plausible to place a Cauchy criterion on $\{T_n\}$ in order to determine whether $sup||T_n|| < \infty$ (obtain results similar to the Banach Steinhauss theorem).

Theorem 3: Let X be a Banach space and Y be a normed linear space. Let $\{T_n\} \subseteq B(X,Y)$ be a Cauchy sequence. Then $sup||T_n|| < \infty$.

Proof: $\{T_n\}$ being Cauchy implies $lim||T_n - T_m|| = 0$. We now have that for each $x \in X$,

$$||T_n x - T_m x|| = ||(T_n - T_m)x|| \quad (1)$$
$$\leq ||T_n - T_m||||x|| \rightarrow 0 \quad (2)$$

This implies that for each $x \in X,\ \{T_n x\}$ is a Cauchy sequence. Hence for each $x \in X,\ \ \{T_n x\}$ is a bounded sequence and as such, there exists $M_x > 0$, such that $||T_n x|| < M_x$.

Now by Theorem 1 (UBT), $sup||T_n|| < \infty$.

Corollary 1: Let X and Y be Banach spaces. Let $\{T_n\} \subseteq B(X,Y)$ be a Cauchy sequence. Then
(i) $sup||T_n|| < \infty$.
(ii) There exists $T \in B(X,Y)$ such that $lim\{T_n x\} = Tx$, for each $x \in X$.
(iii) $||T|| \leq liminf||T_n||$

Proof: (i) follows from Theorem 3. Next, since $\{T_n x\}$ is a Cauchy sequence for each $x \in X$, and Y is complete, then $\{T_n x\}$ converges to $Tx \in Y$. Now, (ii) and (iii) follow from the Banach Steinhauss Theorem.

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Theorem 4: Let X and Y be normed linear spaces. Let $\{T_n\} \subseteq B(X,Y)$ be a Cauchy sequence. Then $sup||T_n|| < \infty$.

Proof: $\{T_n\}$ being Cauchy implies it is a bounded sequence. Therefore, there exists a real constant $K > 0$, such that $||T_n|| < K$, for all $n \geq 1$. Hence, $sup||T_n|| < K$. This completes the proof.

Theorem 5: Let X be a normed linear space and Y be Banach spaces. Let $\{T_n\} \subseteq B(X,Y)$ be a Cauchy sequence. Then
(i) $sup||T_n|| < \infty$.
(ii) There exists $T \in B(X,Y)$ such that $lim\{T_n x\} = Tx$, for each $x \in X$.
(iii) $||T|| \leq liminf||T_n||$

Proof: $\{T_n\}$ being Cauchy implies $lim||T_n - T_m|| = 0$. We now have that for each $x \in X$,

$$\begin{aligned} ||T_n x - T_m x|| &= ||(T_n - T_m)x|| && (3) \\ &\leq ||T_n - T_m||||x|| \ \rightarrow 0 && (4) \end{aligned}$$

This implies that for each $x \in X$, $\{T_n x\}$ is a Cauchy sequence, so that (i) follows either from invoking the UBT or theorem 4 above. Now, since $\{T_n x\}$ is a Cauchy sequence for each $x \in X$, and Y is complete, then $\{T_n x\}$ converges to $Tx \in Y$. Now, (ii) and (iii) follow as in the Banach Steinhauss Theorem.

Remark 1: Our results place conditions on the sequence $\{T_n\}$ itself rather than on the range of $\{T_n\}$, to determine whether $sup||T_n|| < \infty$.

Remark 2: Some of the results presented here (theorems 4 and 5) are in normed linear spaces, and hence do not require very rigorous lemmas such as the Baire's category lemma, for the proofs. Before now, results (Banach Steinhauss Theorem) similar to ours have existed only in Banach spaces, hence our results are of interest.

Remark 3: It is more computationally feasible to evaluate $||T_n - T_m||$ rather than determine whether $\{T_n x\}$ converges to $\{Tx\} \in Y$, for each $x \in X$, (as in the Banach Steinhauss Theorem). Therefore, the conditions in our results are more robust.
Remark 4: Our results (Theorems 4 and 5) are not necessarily consequences of theorem 1 (UBT), and hence are independent.

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